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Symmetries and the Merton portfolio selection model

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Abstract

We revisit the classical Merton portfolio selection model from the perspective of integrability analysis. By an application of a nonlocal transformation the nonlinear partial differential equation for the two-asset model is mapped into a linear option valuation equation with a consumption dependent source term an identical result to that obtained by Cox and Huang using measure theory and stochastic integrals. The nonlinear two-asset equation is then analyzed using Lie symmetry group theory. We show that the linearisation is directly related to the structure of the generalized symmetries.

1 Introduction

A classical financial problem is the modeling of optimal investment-consumption decisions under uncertainty [1, 7, 9, 13, 14]. This was solved in the pioneering work of Merton [13, 14] as an application of dynamic programming and later using approaches based on martingales and stochastic integrals [7, 9]. A dichotomy which arises is that in the Merton dynamic programming result [14] a nonlinear differential equation is derived on the optimal controls and in the Cox-Huang stochastic integrals result [7] a linear differential equation is derived on the optimal controls. The simplest instances of the models [14] and [7] are for a portfolio comprising a stock following a geometric Brownian motion with return, α, and volatility, σ, and a bond with interest rate, r. In [14] this is the nonlinear equation,

$$J_t + (rW - C)J_W - \frac{(\alpha - r)^2}{2\sigma^2}\frac{J_W^2}{J_{WW}} + U(C,t) = 0, \tag{1}$$

while [7] gives the linear equation,

$$F_t + \left(\frac{(\alpha - r)^2}{\sigma^2} - r\right) ZF_Z + \frac{(\alpha - r)^2}{2\sigma^2} Z^2 F_{ZZ} = rF - C(Z,t). \tag{2}$$

The purpose of the present paper is to study the integrability properties of Eq. (1) and to demonstrate that (1) and (2) are equivalent and can be mapped into each other by transformation on the optimal controls. At this point we apply a technique which follows that of Karatzas and Shreve [10] and Carr *et al* [5, 6]. The transformation is implemented by differentiating (1) with respect to W, setting $Z = J_W$ and inverting the roles of W and Z as dependent and independent variables. This provides a way of linking the results directly and goes beyond the explanation of the relationship between the models given in [7]. The transformation is then explored within the context of Lie symmetry groups [3, 12, 16]. The point symmetry structure of (1) was considered in Perets and Yashiv [17] who calculated

the point symmetries of (1) and proved that the solution obtained by Merton [14] for the hyperbolic risk aversion class of utility functions is related to the symmetry based. Here, we show how the generalized infinitesimal Lie symmetries of (1) actually contain (2) inside the symmetry structure and that this structure allows one to construct the linearization. This provides a theoretical basis for our result.

2 The Classical Merton Model

It is assumed that $U(C,t)$ and $V(W,T)$ are given functions. Recall that C is consumption and W total wealth. Furthermore wealth is completely invested in a portfolio comprised of a bond and a stock and there are no exogenous sources of income. The controls which are to be optimized are consumption, C, and the fraction of total wealth invested in each asset: π for the stock and $1-\pi$ for the bond. The underlying market is modeled by assuming that the share price S and the bond price B follow

$$dS_t = \alpha S_t dt + \sigma S_t dZ_t, \tag{3}$$

and

$$dB_t = rB_t dt \tag{4}$$

respectively, and that Z_t is a standard Brownian motion. The parameters α, σ and r are constants. This leads to the budget constraint on wealth [14]

$$dW_t = [(r + (\alpha - r)\pi)W_t - C]dt + \pi\sigma W_t dZ_t. \tag{5}$$

The portfolio optimization problem may then be formulated as

$$J(W,S,t) = \max_{\pi,C}\left[\int_0^t U(C,t)dt + V(W,T)|W_t = W(t), S_t = S(t)\right] \tag{6}$$

subject to the constraints (3) and (5). Since the parameters are constant we may set $J(W,S,t) = J(W,t)$ with an implicit dependence on the share price and the constraint (3) through W. Applying dynamic programming on (5) and (6) we have, [14],

$$J_t + \max_{\pi,C}\left[((r + (\alpha + (\alpha - r)\pi)W - C)J_W + \tfrac{1}{2}\pi^2 W^2\sigma^2 J_{WW}\right] + U(C,t) = 0. \tag{7}$$

The first-order maximization conditions are, [14],

$$U_C = J_W, \tag{8}$$

so that

$$C = U_C^{-1}[J_W] \tag{9}$$

and

$$\pi = -\frac{(\alpha - r)}{\sigma^2}\frac{J_W}{WJ_{WW}} \tag{10}$$

are the optimal controls. Substitution of this into (7) gives the Hamilton-Jacobi-Bellman equation (1) subject to the terminal condition

$$J(W,T) = V(W,T). \tag{11}$$

Note that a nonzero initial wealth, $W(0) = W_0 > 0$, is also required.

3 Reduction of the Nonlinear Merton Equation

We now give the mapping from (1) to (2). It may be seen if one differentiates Eq. (1) totally with respect to W to give

$$J_{tW} + (r - \mu^2)J_W + \left(rW - C + \frac{\partial U}{\partial C}\frac{\partial C}{\partial J_W} - \frac{\partial C}{\partial J_W}J_W\right)J_{WW} + \frac{\mu^2}{2}\frac{J_W^2}{J_{WW}}J_{WWW} = 0, \quad (12)$$

where we have used $C = C(J_W, t)$ and set $\mu^2 = (\alpha - r)^2/\sigma^2$. Although differentiating may seem to complicate the situation it is this step that leads to the linearization. Eq. (12) may be written as the potential system

$$Z = J_W, \quad (13)$$

$$Z_t + (r - \mu^2)Z + (rW - C)Z_W + \frac{\mu^2}{2}\frac{Z}{Z_W^2}Z_{WW} + \frac{\partial C}{\partial Z}Z_W\left[\frac{\partial U}{\partial C} - Z\right] = 0. \quad (14)$$

The optimality condition (8), $U_C = J_W = Z$, gives

$$Z_t + (r - \mu^2)Z + (rW - C)Z_W + \frac{\mu^2}{2}\frac{Z^2}{Z_W^2}Z_{WW} = 0. \quad (15)$$

The problem now is to integrate (15) which may be treated as an independent equation in Z and W. The change of variables

$$W = F(Z, t) \quad (16)$$

inverts the roles of W and Z. The differentials transform as

$$F_t = -\frac{Z_t}{Z_W}, \quad F_Z = \frac{1}{Z_W}, \quad F_{ZZ} = -\frac{Z_{WW}}{Z_W^2}. \quad (17)$$

We note that these are exactly the changes of variables used in [7]. Substitution of (16) and (17 into (15) leads to the linear equation

$$F_t + (\mu^2 - r)ZF_Z + \frac{\mu^2}{2}Z^2F_{ZZ} = rF - C(Z, t). \quad (18)$$

In addition to being identical to (2) this is clearly, apart from the term involving $C(Z, t)$, a form of the Black-Scholes equation [2] in which the left hand side has wealth as an option on marginal utility with a risk premium related to Sharpe's ratio via μ^2 [19] and the right hand side is equivalent to a return on a risk free asset when the consumption strategy is given. It is subject to the intial condition

$$F(Z, 0) = W_0 \quad (19)$$

and the terminal condition

$$F(Z, T) = V_W^{-1}(W, T). \quad (20)$$

The latter follows from $F = W$ and the terminal condtion on (1), $J(W, T) = V(W, T)$, and we have also assumed that V is invertible and differentiable. Finally we remark that this transformation also holds for the time independent, $t \to \infty$, case of the Merton problem as it appears in for example [8].

4 Connection to Lie Symmetry Groups

The transformation we have used above can be explained in terms of Lie symmetry groups for differential equations [3, 16]. These are a well known tools from mathematical physics which have appeared on occasion in various financial contexts [4, 9, 15, 20, 21, 18]. One of the earliest indications of the rôle of symmetries in finance are the comments in the paper by Pliska and Selby [18] which highlight them as a revolutionary way forward in terms

of research. The usual results in the literature are to use point symmetries to derive the transformational properties of given differential equations and to study the integrability properties of models. Largely these have been concerned with local point symmetry properties of the underlying space [4, 20, 21, 15]. This is the calculation in [17] for Eq. (1). Since the transformation we have used depends on differentiation point symmetries do not give sufficient information to lead to (2). A broader class of symmetries are generalized symmetries [16, 3] which expand the space of point symmetries (independent and dependent variables) to include derivatives of the dependent variables. It is these that contain the information relevant for the linearization described in Section 3. We give a brief calculation below.

First we rewrite, for aesthetics reasons, Eq. (1) as

$$u_t + (rx - C)u_x - K\frac{u_x^2}{u_{xx}} + U(C,t) = 0. \tag{21}$$

We are interested in transformations which are functions of the variables (x, t, u, u_x). The corresponding generalized symmetry is a vector field of the form

$$\mathbf{v} = \phi(x,t,u,u_x)\partial_u + \tau(x,t,u,u_x)\partial_t + \xi(x,t,u,u_x)\partial_x. \tag{22}$$

The symmetry condition is

$$\mathbf{v}^{[2]}E = 0, \tag{23}$$

where E is (21) and $\mathbf{v}^{[2]}$ is the second prolongation of the vector field, i.e.

$$\mathbf{v}^{[2]} = \phi\partial_u + \tau\partial_t + \xi\partial_x + \phi^t\partial_{u_t} + \phi^x\partial_{u_x} + \phi^{xx}\partial_{u_{xx}} + \phi^{tt}\partial_{u_{tt}} + \phi^{xt}\partial_{u_{xt}}. \tag{24}$$

Since the equation is independent of u_{xt} and u_{tt} these terms fall away immediately, the remaining terms ϕ^x, ϕ^t, ϕ^{xx} are defined (in the notation of Olver [16]) as

$$\phi^t = D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt}, \tag{25}$$

$$\phi^x = D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{tx}, \tag{26}$$

$$\phi^{xx} = D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{txx}. \tag{27}$$

D_x and D_t are total differential operators. Condition (23) and the optimality condtion (8), now $U_C = u_x$ so $C = C(u_x, t)$, give

$$\phi^t + (rx - C)\phi^x - 2K\frac{u_x}{u_{xx}}\phi^x + K\frac{u_x^2}{u_{xx}^2}\phi^{xx} = 0. \tag{28}$$

The full set of symmetries is not required here. We merely wish to demonstrate how the transformation we obtained is related to the theory. Setting $\phi = \tau = 0$

$$\phi^t = -\xi_t u_x \tag{29}$$

$$\phi^x = -\xi_x u_x \tag{30}$$

$$\phi^{xx} = -\xi_{xx}u_x - 2\xi_x u_{xx}. \tag{31}$$

The symmetry condition is

$$-\xi_t u_x - (rx - C)\xi_x u_x + 2K\frac{u_x}{u_{xx}}\xi_x u_x + r\xi u_x + K\frac{u_x^2}{u_{xx}^2}(-\xi_{xx}u_x - 2\xi_x u_{xx}) = 0. \tag{32}$$

Collecting terms and dividing by u_x

$$-\xi_t - (rx - C)\xi_x - K\frac{u_x^2}{u_{xx}^2} + r\xi = 0. \tag{33}$$

The added constraint $\xi = \xi(t, x, u_x)$ gives

$$-\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial u_x} u_{xt} - (rx - C)\left[\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial u_x} u_{xx}\right] - K\frac{u_x^2}{u_{xx}^2}\left[\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial u_x \partial x} u_{xx} + \frac{\partial^2 \xi}{\partial u_x^2} u_{xx}^2\right] + r\xi = 0. \tag{34}$$

This may be separated in powers of u_{xx} (since ξ is independent of u_{xx}) to give

$$\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial u_x}(2K - r)u_x + Ku_x^2 \frac{\partial^2 \xi}{\partial u_x^2} = r\xi - \frac{\partial \xi}{\partial x}(rx - C) \tag{35}$$

$$\frac{\partial^2 \xi}{\partial u_x \partial x} = 0, \tag{36}$$

$$\frac{\partial^2 \xi}{\partial x^2} = 0. \tag{37}$$

We solve (36) and (37) to give

$$\xi = ax + b \tag{38}$$

where $a = a(t)$ and $b = b(t, u_x)$. Substituting into (35) and separation into powers of x we have

$$\frac{\partial b}{\partial t} + (2K - r)u_x \frac{\partial b}{\partial u_x} + Ku_x^2 \frac{\partial^2 b}{\partial u_x^2} = rb - Ca \tag{39}$$

and

$$\frac{\partial a}{\partial t} = 0. \tag{40}$$

Finally

$$\frac{\partial b}{\partial t} + (2K - r)u_x \frac{\partial b}{\partial u_x} + Ku_x^2 \frac{\partial^2 b}{\partial u_x^2} = rb - a_0 C \tag{41}$$

where

$$a = a_0 \tag{42}$$

for some constant a_0. We have

$$\xi = a_0 x + b(t, u_x), \tag{43}$$

so that the generalized symmetry is

$$\mathbf{v} = a_0 x \partial_x + b(t, u_x)\partial_x. \tag{44}$$

This we write as

$$V_1 = x\partial_x \tag{45}$$

and

$$V_2 = b(t, u_x)\partial_x \tag{46}$$

where without loss of generality $a_0 = 1$.

The information contained in the symmetry can be interpreted from a knowledge of standard symmetry properties. The occurrence of the linear partial differential equation in the function b suggests that a mapping from E to (41) is possible. Since (44) is a generalized symmetry and depends on u_x its structure is related to $D_x E$ rather than E itself, where E is (21) likewise a generalized symmetry as a function containing u_{xx} would depend on $D_x^2 E$ and so forth [16]. This points to the differentiation (12) in Section 3. The change of variables (16) may be seen from using a comparison with the symmetries of the standard one dimensional heat equation [16]. A feature of the point symmetries of a linear partial differential equation such as the one-dimensional heat equation

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial y^2} \tag{47}$$

are the generators

$$G_1 = p\partial_p \tag{48}$$

and

$$G_2 = f(t, y)\partial_p \tag{49}$$

where $f(t, x)$ again satisfies the heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial y^2}. \tag{50}$$

These reflect the linearity and homogeneity of the equation and G_2 in particular is related to the superposition of solutions for linear partial differential equations [3, 16]. Comparing the generators G_1 and G_2 with V_1 and V_2 one may see that the rôles of dependent and independent variables are to be reversed in order to have the symmetries G_2 and V_2 in the same form. This is exactly (16) purely on a Lie group theoretic basis.

5 Conclusion and Remarks

In this paper we have presented a transformation which maps the nonlinear partial differential equation for the classical two-asset Merton portfolio selection model [14, 13] to a linear partial differential equation. Moreover this linear differential equation is identical to the corresponding differential equation in the Cox-Huang model [7] which approaches the classical Merton problem using martingales and stochastic integrals. Our result demonstrates that the integrability of the model is in fact independent of the approach taken and rather has to do with the structure of the differential equation and underlying relationships between contingent claims and portfolio selection. Evidence for this is that the generalized symmetries contain the linear equation derived in [7]. A further point of interest is that the use of symmetry methods and transformations on the evolution equations such as (1) are, from the nature of our result, analogous to the manipulations performed in the martingales and stochastic integrals approach of Cox and Huang [7], before the linear differential equation in their work.

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